Int. J. Heat Mass Transfer. Vol. 26, No. 12, pp. 1847–1856, 1983
Printed in Great Britain

0017-9310/83 $3.00 + 0.00

SHORT-CIRCUIT PHENOMENA: HEAT ENERGY DRAINS IN COMPOSITE CYLINDRICAL SLABS

H. P. YAGODA

Mechanical Engineering Department, Fairleigh Dickinson University, Teaneck, NJ 07666, U.S.A.

(*Received* 19 *January* 1983 *and in revised form* 4 *April* 1983)

Abstract—A basic theoretical analysis of steady heat flow through laminated cylindrical slabs is carried out to introduce the concept of short-circuited diffusion phenomena. It is seen in the case of a two-layer slab that elevated energy drains are a salient characteristic of such phenomena. A limiting case of the solution developed herein, specifically when the conductivity of the lower slab increases beyond all bounds, is used to resolve the long-standing dilemma of discontinuous corner temperatures in the case of the heated semi-infinite strip, or the finite slab. A method to smooth the temperature discontinuities on a physically meaningful basis is developed in this work.

NOMENCLATURE

a	slab radius
A_n, B_n, C_n	Fourier–Bessel coefficients
$F\left(\varepsilon, \frac{W_U}{a}\right)$	the function whose roots define ε
J_0, J_1, J_2	zero, first, and second order Bessel functions of the first kind
k	thermal conductivity
q	normal heat flux component
$Q(\varepsilon)$	steady state heat power for an arbitrarily smoothed boundary temperature
Q_{total}	steady state heat power
$Q_{1\text{-D}}$	one-dimensional steady heat power
r, θ, z	cylindrical coordinates
$S_r\left(\frac{W_U}{a}\right)$	the function given by equation (16b)
T	temperature field
T_i	surface temperature
W	thickness of a slab layer
x, y	rectangular coordinates.

Greek symbols

β_n	roots of J_0
γ_n	roots of J_1
ε	smoothing function: the tapering interval of a discontinuous boundary temperature
ξ_n, ϕ_n	Fourier–Bessel coefficients.

Subscripts

L	lower slab
U	upper slab
1, 2, 3	one of the superimposed solutions.

1. INTRODUCTION

THE INTRINSIC short-circuiting nature of steady heat flow through bodies with discontinuous boundary temperatures has not been recognized previously. The usual case presented most often in the literature and traditionally in university classrooms is that of the semi-infinite strip (or the rectangular slab) with boundary temperatures that are discontinuous at the corners (e.g. refs. [1; 2; 3, pp. 164, 167; 4, pp. 115, 116; 5–8; 9, p. 286]). On physical grounds, in fact, this situation cannot occur: the finite temperature differentials at the corners lead to heat flux singularities. Indeed, it is proved later in this work, that not only is the heat flux singular, but that infinite heat power is required to maintain thermal equilibrium. Of course, this situation is physically unrealistic.

The case of the semi-infinite heated strip described here is an extreme example of the short-circuit phenomena. So that the subsequent discussion of short-circuit phenomena and the ensuing mathematical development does not obscure a practical motivation for the work in this paper, it is noted that a limiting case of the solution for the two-slab model (see Fig. 1) developed herein, specifically when the conductivity of the lower slab increases beyond all bounds, is used to resolve the problem of the semi-infinite cylindrical strip (or the cylindrical slab) in Section 7.

The composite cylindrical slab in Fig. 1 shows a more typical case of the short-circuit phenomena in heat conduction. It is perhaps appropriate to consider briefly the shorted electrical circuit, to fix ideas here about the meaning of the term 'short-circuit'. Webster

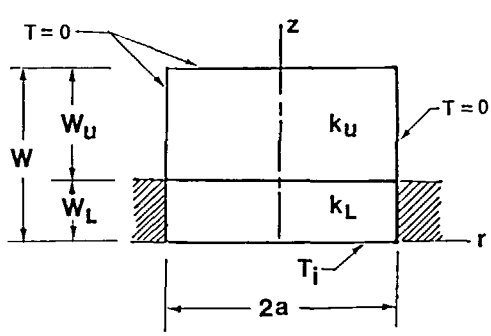

FIG. 1. The laminated cylindrical slab with a short-circuit pathway.

[10] defines short-circuit as a bypass, or as "a connection of comparatively low resistance accidentally or intentionally made between points on a circuit between which the resistance is normally much greater". Thus, the lower slab shown in Fig. 1 is configured so that a short-circuiting pathway is generated, radially outward across the lower cylindrical surface of the upper slab, when the conductivity k_L of the lower slab becomes large, i.e. in the case when $k_L/k_U \gg 1$. The principal direction of heat energy flow, upward along the axial direction of the upper slab, is thereby short-circuited in favor of the aforementioned lateral bypass.

Diffusion through laminated structures is a process of central importance in a number of fields of wide diversity; for example, the conduction of heat along clad nuclear reactor rods [11], or the diffusion of mass through laminated membranes [12]. Although two-dimensional heat conduction problems have received considerable attention recently, e.g. refs. [13, 14], short-circuiting pathways have not been considered previously. This situation may stem from the preponderant emphasis in the literature given to the development of solutions for temperature distributions, often by approximate methods [3, 11, 13, 15, 16].

Thus, it may be appropriate to underscore here, prior to the ensuing mathematical development, that the overall heat power required to maintain the steady state process, rather than the temperature distribution, is the key quantity of interest in the short-circuiting phenomena.

2. FORMULATION OF THE CONDUCTION PROBLEM WITH A SHORT-CIRCUIT

As remarked earlier, the laminated slab shown in Fig. 1 is configured so that a short-circuiting pathway is generated when the conductivity of the lower slab becomes large. The governing diffusion equations for the axisymmetric temperature fields $T_L(r, z)$ and $T_U(r, z)$, in the lower and upper cylindrical slabs, respectively, under steady-state conditions in isotropic solids, are as follows:

$$\frac{\partial^2 T_L}{\partial r^2} + \frac{1}{r}\frac{\partial T_L}{\partial r} + \frac{\partial^2 T_L}{\partial z^2} = 0, \quad \text{for} \quad 0 < r < a, \quad 0 < z < W_L,$$

$$\frac{\partial^2 T_U}{\partial r^2} + \frac{1}{r}\frac{\partial T_U}{\partial r} + \frac{\partial^2 T_U}{\partial z^2} = 0, \quad \text{for} \quad 0 < r < a, \quad W_L < z < W. \tag{1}$$

The conditions imposed on the boundaries of the slab are:

$$T_L(r, 0) = T_i$$

$$\left.\begin{aligned} &\frac{\partial T_L}{\partial r}(a, z) = 0 \\ &\text{bounded } T_L(0, z) \end{aligned}\right\} \quad \text{for} \quad 0 < z < W_L,$$

$$T_U(r, W) = 0$$

$$\left.\begin{aligned} &T_U(a, z) = 0 \\ &\text{bounded } T_U(0, z) \end{aligned}\right\} \quad \text{for} \quad W_L < z < W. \tag{2}$$

In accordance with energy considerations, the following interface equations further constrain the temperature fields:

$$T_L(r, W_L) = T_U(r, W_L),$$

$$k_L \frac{\partial T_L}{\partial z}(r, W_L) = k_U \frac{\partial T_U}{\partial z}(r, W_L). \tag{3}$$

Utilizing the principle of superposition, the formulation given by equations (1)–(3) is simplified in the next section.

3. ANALYSIS OF THE TWO-LAYER LAMINATED CYLINDRICAL SLAB

It is well known that the linearity of the Laplace equation associated with several linear nonhomogeneous boundary conditions may be separated into sets of Laplace equations, each associated with only one nonhomogeneous condition. Thus, the problem formulated in the previous section is divided here (see Fig. 2) into three separate formulations, which are inter-related through the interface conditions (3). When

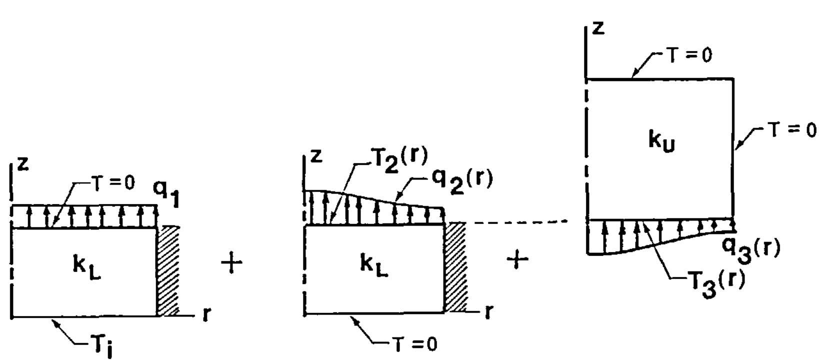

FIG. 2. Superposition of formulations.

superimposed, the axisymmetric temperature fields $T_1(z)$ and $T_2(r, z)$, which both pertain to diffusion in the lower slab, must sum to the temperature distribution $T_L(r, z)$:

$$T_1(z)+T_2(r, z) = T_L(r, z). \tag{4}$$

The solution to the first formulation shown in Fig. 2 is one-dimensional and may be written in a straightforward manner, thus,

$$T_1(z) = T_i\left(1-\frac{z}{W_L}\right), \quad q_1 = \frac{k_L}{W_L}T_i. \tag{5}$$

Both the heat flux q_1 and the temperature $T_1(z)$ are uniform with respect to the radial coordinate r. The formulations for the second and third problems shown in Fig. 2 consist of the Laplace equations and following boundary conditions:

$$\begin{aligned} &T_2(r, 0) = 0 \\ &\left.\begin{aligned}&\frac{\partial T_2}{\partial r}(a, z) = 0 \\ &\text{bounded } T_2(0, z)\end{aligned}\right\} \text{ for } 0 < z < W_L, \\ &T_2(r, W_L) = T_2(r) \\ &-k_L\frac{\partial T_2}{\partial z}(r, W_L) = q_2(r) \\ &T_U(r, W) = 0 \\ &\left.\begin{aligned}&T_U(a, z) = 0 \\ &\text{bounded } T_U(0, z)\end{aligned}\right\} \text{ for } W_L < z < W. \\ &T_U(r, W_L) = T_3(r) \\ &-k_U\frac{\partial T_U}{\partial z}(r, W_L) = q_3(r) \end{aligned} \tag{6}$$

Note that $T_3(r)$ and $q_3(r)$ are the interface temperature and heat flux, respectively. These functions are related to the corresponding quantities $T_2(r)$ and $q_2(r)$ through the interface conditions (3) and equation (5), namely

$$T_3(r) = T_2(r), \tag{7a}$$

$$q_3(r) = \frac{k_L}{W_L}T_i+q_2(r). \tag{7b}$$

Employing the method of separation of variables, solutions that satisfy the homogeneous conditions of equations (6) are seen to have the following form:

$$T_2(r, z) = A_0 z+\sum_{n=1}^{\infty} A_n \sinh\left(\gamma_n\frac{z}{a}\right)J_0\left(\gamma_n\frac{r}{a}\right), \tag{8a}$$

$$\frac{\partial T_2}{\partial z}(r, z) = A_0+\frac{1}{a}\sum_{n=1}^{\infty} A_n\gamma_n \cosh\left(\gamma_n\frac{z}{a}\right)J_0\left(\gamma_n\frac{r}{a}\right), \tag{8b}$$

$$T_3(r, z) = \sum_{m=1}^{\infty} B_m \sinh\frac{\beta_m}{a}(z-W)J_0\left(\beta_m\frac{r}{a}\right), \tag{9a}$$

$$\frac{\partial T_3}{\partial z}(r, z) = \frac{1}{a}\sum_{m=1}^{\infty} B_m\beta_m \cosh\frac{\beta_m}{a}(z-W)J_0\left(\beta_m\frac{r}{a}\right), \tag{9b}$$

where β_m and γ_n are roots of the zero and first order Bessel functions of the first kind [17], J_0 and J_1, respectively. In view of the smooth conditions that are imposed on the slab surfaces, i.e. equations (2), one can assume here that each of the interface functions $T_2(r)$ and $q_2(r)$ may be written as a Bessel series:

$$T_2(r) = W_L A_0+\sum_{n=1}^{\infty}\xi_n J_0\left(\gamma_n\frac{r}{a}\right),$$

$$q_2(r) = -k_L A_0+\sum_{n=1}^{\infty}\phi_n J_0\left(\gamma_n\frac{r}{a}\right). \tag{10}$$

Comparing equations (10) and (8), it is seen that the coefficients ξ_n and ϕ_n must be related, and specifically to the undetermined coefficients A_n, in order that all conditions are satisfied for heat flow through the lower slab, namely, the conditions on $T_2(r, z)$:

$$\xi_n = A_n \sinh\left(\gamma_n\frac{W_L}{a}\right), \quad \phi_n = -\frac{\gamma_n}{a}k_L A_n \cosh\left(\gamma_n\frac{W_L}{a}\right), \quad \text{for } n \geqslant 1. \tag{11}$$

Thus, the infinite number of quantities A_n may be considered as the coefficients of the interface temperature and heat flux functions. The values of these coefficients are to be determined so that all of the conditions on the heat flow through the upper slab are satisfied, i.e. the conditions on $T_3(r, z)$, specifically equations (6) and (9). It is now advantageous to rewrite equations (10) in the form of equations (9) by employing the following Fourier–Bessel expansion [4, p. 164]:

$$J_0\left(\gamma_n\frac{r}{a}\right) = \sum_{m=1}^{\infty} C_m J_0\left(\beta_m\frac{r}{a}\right), \tag{12a}$$

where

$$C_m = \frac{2}{a^2J_1^2(\beta_m)}\int_0^a rJ_0\left(\gamma_n\frac{r}{a}\right)J_0\left(\beta_m\frac{r}{a}\right)\,dr, \tag{12b}$$

for $0 \leqslant r < a$. If equations (10)–(12) are substituted into equations (7), the following series are obtained for the unknown interface temperature and heat flux:

$$T_3(r) = W_L A_0+\sum_{m=1}^{\infty}\left\{\sum_{n=1}^{\infty} A_n C_m \times \sinh\left(\gamma_n\frac{W_L}{a}\right)\right\}J_0\left(\beta_m\frac{r}{a}\right),$$

$$q_3(r) = \frac{k_L T_i}{W_L}-k_L A_0-\frac{k_L}{a}\sum_{m=1}^{\infty} \times\left\{\sum_{n=1}^{\infty} A_n C_m\gamma_n \cosh\left(\gamma_n\frac{W_L}{a}\right)\right\}J_0\left(\beta_m\frac{r}{a}\right). \tag{13}$$

Note that the Bessel terms of equations (13) have precisely the same form as those of equations (9); therefore, the bracketed coefficients of the Bessel series in equations (13) bear a relationship similar to equation (11), in order that all of the conditions are satisfied for flow of heat through the upper slab. If the solution for the temperature distribution were a primary motivation, which it is not, a rather intractable infinite number of simultaneous equations for the coefficients A_n may be developed: the constant terms on the RHS of equations (13) would first be written as Fourier–Bessel expansions in terms of $J_0[\beta_m(r/a)]$. However, central emphasis in this work is directed to a determination both of the heat power needed to sustain the steady flow of energy through the slab and the relation between elevated heating requirements and the thermal conductivities of the laminated components.

In this regard, it may be appropriate to recall here that a salient characteristic of steady state heat conduction problems is the flow of energy into a receptive area of the bounding surface of a body, which then exits, without internal accumulation, through an expulsive region of the body's surface. Often, the temperature over the receptive and expulsive regions of the surface are fixed by overall considerations of the ongoing energy process. Thus, the key quantity is, from a practical viewpoint, the heat power needed to sustain the thermal gradients across the body. It follows directly from the divergence theorem and Laplace's equation that the heat power across the interface of the cylinders is precisely synonomous with this overall heat power.

In accordance with the foregoing discussion, a formula is sought for computing the equilibrium heat power, Q_{total}:

$$Q_{\text{total}} = \int_0^a \int_0^{2\pi} q_3(r) r \, dr \, d\theta = 2\pi \int_0^a r q_3(r) \, dr. \quad (14)$$

Employing superposition and axial symmetry, i.e. equation (7b) and also equation (8b), equation (14) can be rewritten in the following form:

$$\begin{aligned} Q_{\text{total}} &= \pi a^2 \frac{k_L T_i}{W_L} + 2\pi \int_0^a r q_2(r) \, dr, \\ &= \pi a^2 \left(\frac{k_L}{W_L} T_i - k_L A_0 \right) - \frac{2\pi}{a} k_L \sum_{n=1}^{\infty} \\ &\quad \times A_n \gamma_n \cosh\left(\gamma_n \frac{W_L}{a} \right) \int_0^a r J_0\left(\gamma_n \frac{r}{a} \right) dr. \end{aligned} \quad (15)$$

Note that the integral in equation (15) is zero for all $n \geqslant 1$. It is concluded, therefore, that none of the A_n terms contribute to the average heat flux on the interface. In a similar manner, it can be seen that the A_n terms do not contribute to the average interface temperature, $W_L A_0$, e.g. equations (8a) and (13). Thus, it may be concluded that the average interface temperature effectively stems from the constant heat flux terms in equation (13).* The following relation between these heat flux terms and the average interface temperature is developed in the Appendix by considering heat flow in the upper slab:

$$W_L A_0 = \frac{4a}{k_U} \left(\frac{k_L T_i}{W_L} - k_L A_0 \right) S_r\left(\frac{W_U}{a} \right), \quad (16a)$$

* Note that the Fourier–Bessel series in equations (13) so modify the constant terms that the interface temperature will satisfy the boundary condition on the cylindrical surface of the slab.

where

$$S_r\left(\frac{W_U}{a} \right) = \sum_{n=1}^{\infty} \frac{1}{\beta_n^3} \tanh\left(\beta_n \frac{W_U}{a} \right). \quad (16b)$$

One can determine A_0 from equation (16a), thus

$$A_0 = \left(\frac{T_i}{W_L} \right) \frac{S_r(W_U/a)}{\frac{1}{4}(k_U/k_L)(W_L/a) + S_r(W_U/a)}. \quad (17)$$

Substituting A_0 into equation (15) gives the heat power required to maintain the temperature gradient T_i across the slab:

$$\frac{Q_{\text{total}}}{k_U T_i} = \pi a \frac{(k_L/k_U)}{(W_L/a)} \times \left\{ 1 - \frac{1}{1 + \{\frac{1}{4}[(k_U/k_L)(W_L/a)]/[S_r(W_U/a)]\}} \right\}. \quad (18)$$

The analogous heat power expression for one-dimensional diffusion through composite media is well known:

$$\left. \frac{Q_{\text{total}}}{k_U T_i} \right|_{1\text{-D}} = \frac{\pi a}{(k_U/k_L)(W_L/a) + (W_U/a)}. \quad (19)$$

In view of the foregoing multi-phased mathematical development, it is corroborative to note here that equation (18) reduces, as it properly should, to equation (19) in all legitimate cases of the geometric and thermal parameters; i.e. specifically as $W_U/a \to 0$, or $k_U \to \infty$, or $k_U \to 0$, or when both $W_U/a \to 0$ simultaneously as $W_L/a \to 0$. In the last of these limiting cases, $S_r(W_U/a) \to \frac{1}{4}(W_U/a)$, which follows from the summation

$$\sum_{n=1}^{\infty} \beta_n^{-2} = \tfrac{1}{4}. \quad (20)$$

Summation (20) converges very slowly, indeed. Employing a PDP-11 FORTRAN algorithm, sums obtained by including 1000, 2000, 3000 and 4000 terms of the series are 0.24990, 0.24995, 0.24997 and 0.24997, respectively.

Values of $S_r(W_U/a)$ were computed for $0 < W_U/a < 10$ and are plotted in Fig. 3 and listed in Table 1 for typical values of W_U/a. It is noted that the derivation in this work does not hold precisely when $W_L/a = 0$, e.g. equation (7b).

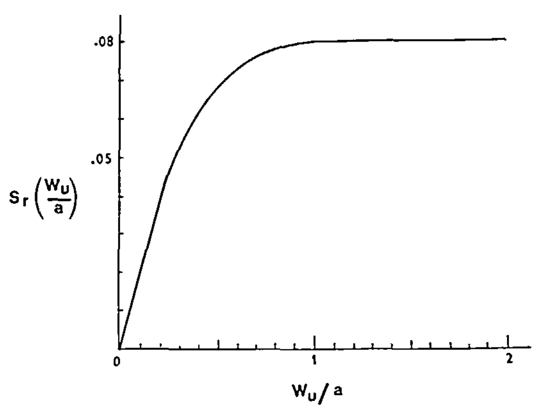

FIG. 3. The summation in equation (16b).

Table 1. The summation in equation (16b)

$\frac{W_U}{a}$	$S_r\left(\frac{W_U}{a}\right)$
0	0
0.1	0.02237
0.2	0.03984
0.3	0.05296
0.4	0.06241
0.5	0.06893
0.6	0.07326
0.7	0.07608
0.8	0.07788
0.9	0.07901
1	0.07972
2	0.08087
3	0.0808812
4	0.0808813
⋮	⋮
∞	0.0808813

4. DISCUSSION AND A COMPARISON FOR THE TWO-LAYER LAMINATED CYLINDRICAL SLAB

Equation (18) relates the equilibrium heat power to both the thicknesses and thermal conductivities of the laminated slab shown in Fig. 1. Figure 4, constructed on the basis of equation (18), shows elevated heating power as the conductivity of the underslab increases. It is seen from Fig. 4 that highly conductive underslabs provide a very substantive lateral bypass for the short-circuited flow of heat energy.

To quantify the degree of the short circuit, the heat power here is compared with the one-dimensional case in which there cannot be a short-circuited pathway, i.e. when the cylindrical surfaces in Fig. 1 are entirely insulated. This comparison, shown in Fig. 5, is obtained by dividing equation (18) by equation (19):

$$\frac{Q_{\text{total}}}{Q_{1\text{-D}}} = \frac{1+(k_L/k_U)(W_U/W_L)}{1+[4(k_L/k_U)S_r(W_U/a)/(W_L/a)]}. \qquad (21)$$

Figure 5 shows that for a constant surface temperature differential T_i, the heat power drain ratio increases sharply with underslab conductivity, and by a factor of 2 or 3 for the value of the parameters selected. The heat power drain levels off, however, above a critical range of the slab conductivity ratio, k_L/k_U. Thus, the short-circuiting effect is fully established above this range, and further increases in underslab conductivity have no appreciable effect.

Figure 5 also shows that lowered underslab conductivity, i.e. $k_L/k_U \ll 1$, will stem the drain of heat power. In fact, as $k_L/k_U \to 0$, the heat power in the case having a short-circuit pathway is no larger than the fully-insulated one-dimensional case. It may be concluded, therefore, that an underslab of lowered conductivity acts as a short-circuit plug.

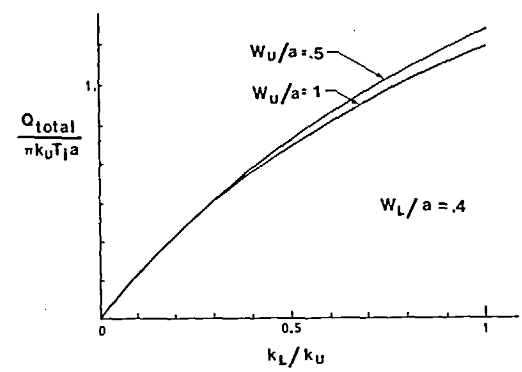

FIG. 4(a). Non-dimensional heat power vs slab conductivity ratio.

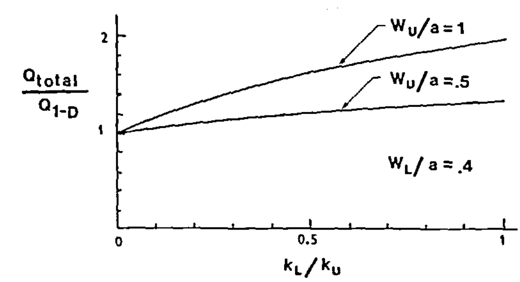

FIG. 5(a). Ratio of short-circuited to non-short-circuited heat power.

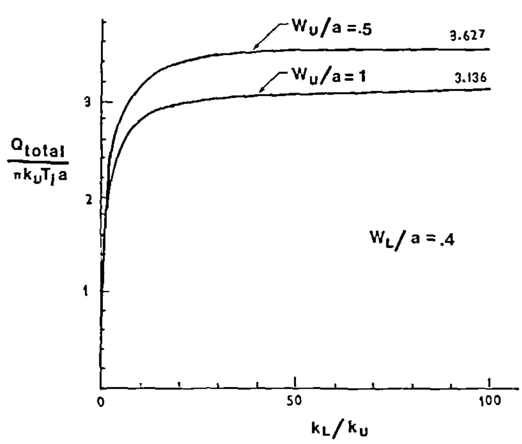

FIG. 4(b). Non-dimensional heat power vs slab conductivity ratio.

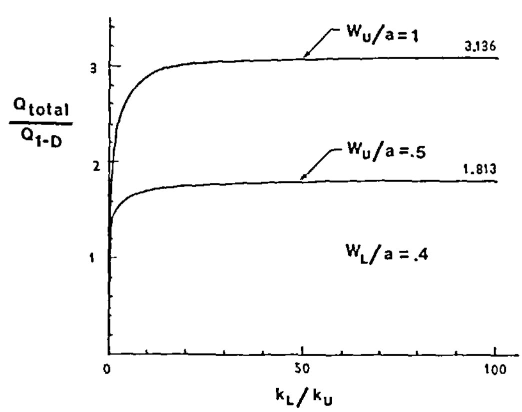

FIG. 5(b). Ratio of short-circuited to non-short-circuited heat power.

5. AN EXTREME CASE OF SHORT-CIRCUIT PHENOMENA

The semi-infinite strip, heated as shown in Fig. 6, has been considered in the literature as early as the mid-nineteenth century. It was in 1878 that Fourier's solution for the temperature distribution was published, apparently to show an application of the series that bears his name [2, pp. 131–155]. This same problem, or the analogous finite rectangular slab, has been historically presented thereafter in the literature on heat conduction [3, pp. 164, 167; 4, pp. 115, 116; 5–7]. What has not been heretofore recognized, however, is the intrinsically short-circuit nature of these problems. Indeed, the corner temperature discontinuities lead to infinite energy rates, as will be proved subsequently.

Carslaw and Jaeger [3, p. 164] give a closed form solution for the temperature field in the semi-infinite strip (see Fig. 6):

$$T(x, y) = \frac{2}{\pi} \arctan \left(\frac{\sin (\pi/L)x}{\sinh (\pi/L)y}\right). \tag{22}$$

Equation (22) allows an easy derivation of the heat power. It is a straightforward matter, thereby, to write the heat flux on the boundary $y = 0$:

$$q(x) = \frac{2k}{L} \frac{1}{\sin^2(\pi/L)x}. \tag{23}$$

Integration over the interval $\varepsilon \leqslant (x/L) \leqslant 1-(\varepsilon/L)$ on the boundary $y = 0$ gives the heat power, $Q(\varepsilon)$:

$$Q(\varepsilon) = \int_{\varepsilon L}^{L-\varepsilon L} q(x)\, dx = \frac{4}{\pi} k \cot (\pi\varepsilon). \tag{24}$$

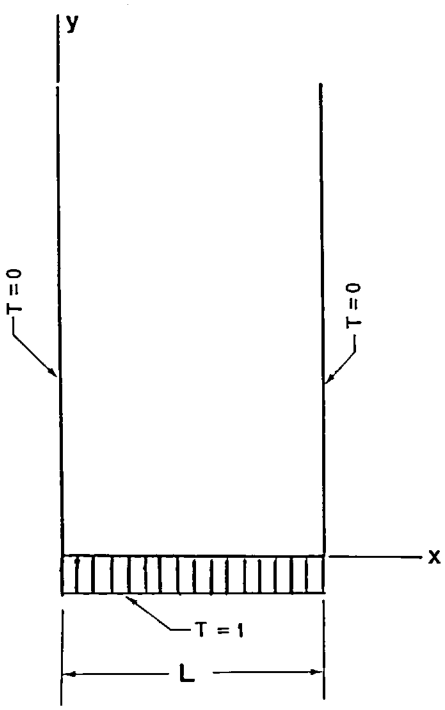

FIG. 6. Discontinuous temperature on the boundary of the semi-infinite strip.

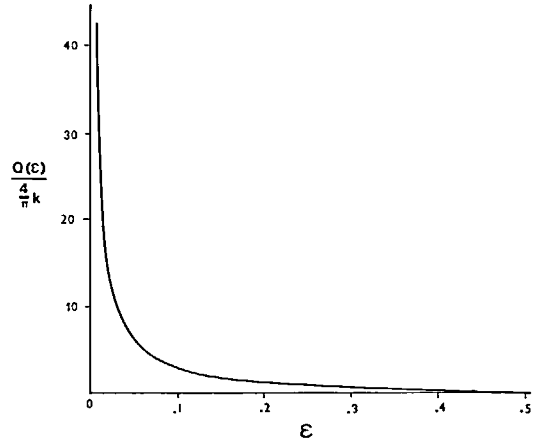

FIG. 7. The heat power over the interval $\varepsilon L < x < (1-\varepsilon)L$ on the boundary of the semi-infinite strip.

$Q(\varepsilon)$ is shown in Fig. 7 for the physically meaningful range $0 < \varepsilon \leqslant 0.5$. Clearly, the heat power grows beyond all bounds as $\varepsilon \to 0$.

It is obvious now that problems of this type are extreme cases of the short-circuit phenomena; thus, smoothing of the boundary discontinuities appears to be in order. However, the extent of smoothing (i.e. the value of ε in Fig. 8) cannot be inferred from Fig. 7, since the curve does not asymptotically approach a positive limit as ε increases. To resolve this dilemma, a method for smoothing the boundary temperature discontinuity on a physically meaningful basis is developed in Section 7.

6. SOLUTIONS FOR ARBITRARY SMOOTHING OF THE BOUNDARY TEMPERATURE DISCONTINUITY

Consider the cylindrical slab shown in Fig. 8. In accordance with the foregoing discussion, the boundary temperature has been smoothed by a parabolically-shaped taper. Note, however, that the smoothing interval εa is as yet arbitrary. The value of ε

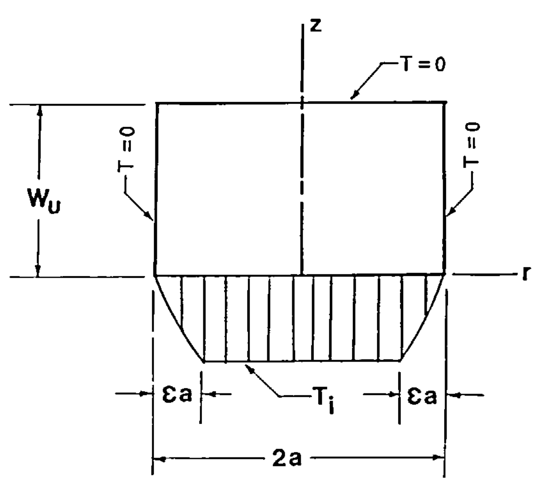

FIG. 8. Smoothed temperature on the boundary of the cylindrical slab.

will be determined in Section 7. The governing diffusion equation for the axisymmetric temperature field $T(r, z)$, under steady-state conditions in an isotropic solid, is:

$$\frac{\partial^2 T}{\partial r^2}+\frac{1}{r}\frac{\partial T}{\partial r}+\frac{\partial^2 T}{\partial z^2}=0,$$
$$\text{for} \quad 0<r<a, \quad 0<z<W_U. \tag{25}$$

Conditions imposed on the boundaries of the slab are:

$$T(r,0)=T_i(r)=\begin{cases} T_i, & \text{for} \quad 0<\dfrac{r}{a}<1-\varepsilon, \\ T_i\dfrac{[1-(r^2/a^2)]}{[1-(1-\varepsilon)^2]}, & \text{for} \quad \dfrac{r}{a}\geqslant 1-\varepsilon, \end{cases}$$
$$T(r, W_U)=0, \tag{26}$$
$$T(a,z)=0,$$
$$\text{bounded}\ \frac{\partial T}{\partial r}(0,z).$$

As mentioned earlier in the analysis of the two-slab case, solutions for the homogeneous conditions of equations (26) are seen to have the following form, where β_n are roots of the zero order Bessel function of the first kind, J_0:

$$T(r,z)=\sum_{n=1}^{\infty} C_n \sinh\frac{\beta_n}{a}(W_U-z)\,J_0\left(\beta_n\frac{r}{a}\right). \tag{27}$$

To develop the solution for the temperature field, the Fourier–Bessel series is obtained for the function $T_i(r)$, in the interval $0<r<a$, by integrating by parts the usual formula for the coefficients, e.g. ref. [4, p. 163]:

$$T_i(r)=\frac{4T_i}{[1-(1-\varepsilon)^2]}\sum_{n=1}^{\infty}\times\frac{J_2(\beta_n)-(1-\varepsilon)^2J_2[\beta_n(1-\varepsilon)]}{\beta_n^2J_1^2(\beta_n)}J_0\left(\beta_n\frac{r}{a}\right). \tag{28}$$

The remaining non-homogeneous condition of equations (26) will be satisfied if

$$C_n=\frac{4T_i}{[1-(1-\varepsilon)^2]}\frac{\{J_2(\beta_n)-(1-\varepsilon)^2J_2[\beta_n(1-\varepsilon)]\}}{\beta_n^2J_1^2(\beta_n)\sinh[\beta_n(W_U/a)]}. \tag{29}$$

Thus, it is a straightforward matter to write the solution for the temperature field, and by employing Fourier's law of heat conduction, the heat flux $q(r)$ on the boundary $z=0$:

$$T(r,z)=\frac{4T_i}{[1-(1-\varepsilon)^2]}\sum_{n=1}^{\infty}\frac{J_2(\beta_n)-(1-\varepsilon)^2J_2[\beta_n(1-\varepsilon)]}{\beta_n^2J_1^2(\beta_n)\sinh[\beta_n(W_U/a)]}\times\sinh\frac{\beta_n}{a}(W_U-z)J_0\left(\beta_n\frac{r}{a}\right), \tag{30}$$

$$q(r)=\frac{4k_UT_i}{[1-(1-\varepsilon)^2]a}\sum_{n=1}^{\infty}\frac{J_2(\beta_n)-(1-\varepsilon)^2J_2[\beta_n(1-\varepsilon)]}{\beta_nJ_1^2(\beta_n)\tanh[\beta_n(W_U/a)]}\times J_0\left(\beta_n\frac{r}{a}\right). \tag{31}$$

Integrating equation (31), one obtains the following heat power equation:

$$Q(\varepsilon)=\int_0^a\int_0^{2\pi} rq(r)\,dr\,d\theta$$
$$=2\pi\int_0^a rq(r)\,dr=\frac{8\pi k_UT_ia}{[1-(1-\varepsilon)^2]}\times\sum_{n=1}^{\infty}\frac{J_2(\beta_n)-(1-\varepsilon)^2J_2[\beta_n(1-\varepsilon)]}{\beta_n^2J_1(\beta_n)\tanh[\beta_n(W_U/a)]}, \tag{32}$$

where the known definite integral

$$\int_0^a rJ_0\left(\beta_n\frac{r}{a}\right)dr=\frac{a^2}{\beta_n}J_1(\beta_n), \tag{33}$$

is employed to obtain equation (32). Note that the heat power $Q(\varepsilon)$ increases beyond all bounds as $\varepsilon\to 0$. The singular nature of $Q(\varepsilon)$ is shown in Fig. 9 for two selected aspect ratios, specifically $W_U/a=1$ and 10. To this point, however, it is noted that the value of ε is arbitrary and, perhaps more significantly, the heat power is indeterminate.

7. NATURAL SMOOTHING OF THE BOUNDARY TEMPERATURE DISCONTINUITY

The purpose at hand here is not only to render the heat power $Q(\varepsilon)$ determinate, but to do so on a physically meaningful basis. Therefore, one seeks to relate the smoothed boundary temperature of Fig. 8 to a situation in which the slab undersurface is maintained entirely at a constant temperature T_i, but without a corner temperature discontinuity. These requirements are precisely fulfilled by a limiting case of the solution developed earlier in this work, for the two-layer laminated slab shown in Fig. 1, when the conductivity of the lower slab increases beyond all bounds.

Perspective on the relevance of the aforementioned limiting case ($k_L\to\infty$) of the laminated slab may

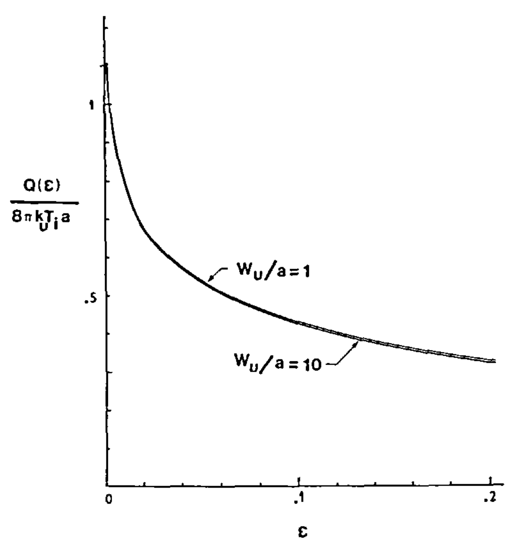

FIG. 9. The heat power for arbitrarily smoothed temperature on the boundary of the cylindrical slab.

perhaps be enhanced by considering the case of one-dimensional steady heat flow through the underslab. For this case, the relation between the heat flux q and the thermal gradient $(T_i - T_0)$ across the underslab is well known:

$$q = \frac{k_L}{W_L}(T_i - T_0), \quad \text{or} \quad T_0 = T_i - \frac{W_L}{k_L} q. \tag{34}$$

It may be observed from equations (34) that the temperature field becomes uniform throughout the underslab volume, i.e. $T_0 \to T_i$, for highly conductive underslabs, i.e. when $k_L \to \infty$. One concludes, therefore, that both the form and magnitude of the square-wave boundary temperature T_i, on the lower surface of the underslab, is in this case transmitted undisturbed to the upper surface. Thus, it may be argued that a superconductive underslab will most evenly transmit the constant form of the boundary temperature T_i to the interface of the composite cylindrical slab shown in Fig. 1.

In accordance with the foregoing discussion, the interface temperature of the laminated slab, when $k_L \to \infty$, will be quite similar to the smoothed boundary temperature shown in Fig. 8. It is noted in this limiting case, that the interface temperature is smoothed physically, hereafter called natural smoothing, by diffusion of heat flowing through the superconductive lower slab. Thus, by equating the heat power in each of the cases just described, both the heat power $Q(\varepsilon)$ [equation (32)] and the smoothing function ε are rendered determinate. An exact solution for the heat power required to sustain steady conduction through composite slabs is given by equation (18). In the case when the underslab conductivity is very large, i.e. when $k_L/k_U \to \infty$, equation (18) reduces, after some mathematical manipulation, to

$$Q_{\text{total}} = \frac{\pi a k_U T_i}{4 S_r(W_U/a)}. \tag{35}$$

Table 2. The smoothing taper ε of the temperature discontinuity on the boundary of the cylindrical slab

$\frac{W_U}{a}$	ε	Percent error*
0.05	0.01747	0.00081
0.10	0.03359	0.00014
0.15	0.04830	0.00150
0.20	0.06153	0.00004
0.30	0.08351	0.00014
0.40	0.09972	0.00014
0.50	0.11102	0.00024
0.60	0.11855	0.00038
0.70	0.12344	0.00020
0.80	0.12655	0.000025
1.0	0.12973	0.00015
1.5	0.13155	0.00057
2	0.13171	0.00010
10	0.131727	0.00013
50	0.131727	0.00013

* See equation (37).

In view of the preceding remarks, equation (35) fixes the magnitude of the smoothed heat power. The smoothing function $\varepsilon(W_U/a)$ can now be determined from the following transcendental equation, obtained by equating equations (32) and (35):

$$F\left(\varepsilon, \frac{W_U}{a}\right) = \frac{1}{[1-(1-\varepsilon)^2]} \sum_{n=1}^{\infty} \times \frac{(2/\beta_n)J_1(\beta_n) - (1-\varepsilon)^2 J_2[\beta_n(1-\varepsilon)]}{\beta_n^2 J_1(\beta_n) \tanh[\beta_n(W_U/a)]} - \frac{1}{32 S_r(W_U/a)} = 0. \tag{36}$$

A PDP-11 FORTRAN algorithm based on the secant method [9, p. 71] was used to obtain the roots $\varepsilon(W_U/a)$ of equation (36).

The errors, which are defined as follows and listed in Table 2, indicate that the values of ε so obtained have been computed to good engineering accuracy:

$$\text{percent error} = 32 S_r\left(\frac{W_U}{a}\right) \left| F\left(\varepsilon, \frac{W_U}{a}\right) \right| \times 100. \tag{37}$$

The smoothing function, $\varepsilon(W_U/a)$, is plotted in Fig. 10. Values of the Bessel functions J_0, J_1, and J_2 were computed employing a FORTRAN algorithm based on both series and asymptotic representations, e.g. refs. [17; 18, p. 139].

8. SUMMARY AND CONCLUSIONS

Both infinite heat flux and infinite heating power stem from a discontinuous boundary temperature on contiguous surface regions of a body in thermal equilibrium. This situation, although unrealistic only if the discontinuity is not smoothed, is one of

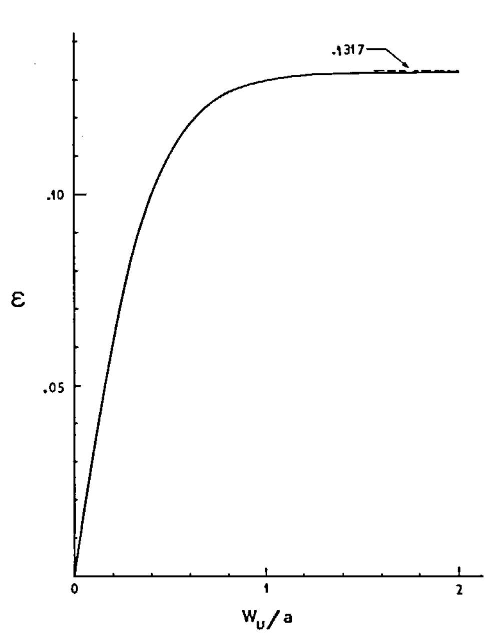

FIG. 10. Natural smoothing depends on the slab aspect ratio.

considerable practical importance and therefore, is resolved herein on a physical basis.

It may be argued in the case just described that realistic smoothing of boundary temperature discontinuities should be based on a consideration of the interaction of the energy source with the heated body, in view of the infinite energy rate which would otherwise be a mathematical consequence. Precisely this approach is taken herein. The single-layer slab with discontinuous corner temperature is extended to a limiting case of a two-layer laminated slab: the lower slab may be considered as part of the energy source.

A natural smoothing function ε is obtained in this manner. Thereby, the dilemma of infinite energy rate is resolved in the practical case when contiguous surface regions of a cylindrical body in thermal equilibrium are maintained at essentially constant but different temperatures. The taper of the discontinuity so obtained represents a lowerbound, because the superconductive underslab employed most evenly spreads the heat from the energy source to the slab interface. Thus, the smoothing will be minimal and is synonymous with the best approximation to a boundary temperature of square-wave form that can be realized under practical conditions.

Acknowledgements—The author acknowledges with gratitude both the assistance by personnel and use of the computational facilities of the Department of Electrical Engineering at Fairleigh Dickinson University.

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APPENDIX

It is a straightforward matter to relate the average temperature on the upper slab boundary $z = W_L$ (see Fig. 2) to a uniform heat flux q_0 imposed on the same boundary. The form of the solution is given by equations (9). If q_0 is expanded in a Fourier–Bessel series, thus

$$q_0 = 2q_0 \sum_{m=1}^{\infty} \frac{J_0[\beta_m(r/a)]}{\beta_m J_1(\beta_m)}, \tag{A1}$$

the following values of the coefficients B_m are obtained by a term by term comparison of equations (A1) and (9b):

$$B_m = -\frac{2q_0 a}{k_U \beta_m^2 J_1(\beta_m)\cosh[\beta_m(W_U/a)]}. \tag{A2}$$

The temperature on the boundary then follows by substituting equation (A2) into equation (9a):

$$T(r, W_L) = \frac{2q_0 a}{k_U} \sum_{m=1}^{\infty} \frac{\tanh[\beta_m(W_U/a)]}{\beta_m^2 J_1(\beta_m)} J_0\left(\beta_m \frac{r}{a}\right). \tag{A3}$$

If equation (A3) is now integrated, the desired relation between average boundary temperature and heat flux q_0 is obtained:

$$\begin{aligned} T_{AV} &= \frac{1}{\pi a^2}\int_0^a \int_0^{2\pi} rT(r, W_L)\,dr\,d\theta, \\ &= \frac{4q_0}{ak_U}\sum_{m=1}^{\infty}\frac{\tanh[\beta_m(W_U/a)]}{\beta_m^2 J_1(\beta_m)} \\ &\quad \times \int_0^a rJ_0\left(\beta_m\frac{r}{a}\right) dr, \\ &= \frac{4aq_0}{k_U}\sum_{m=1}^{\infty}\frac{1}{\beta_m^3}\tanh\left(\beta_m\frac{W_U}{a}\right). \end{aligned} \tag{A4}$$

PHENOMENE DE COURT-CIRCUIT: DRAINAGE DE L'ENERGIE THERMIQUE DANS DES COUCHES CYLINDRIQUES COMPOSITES

Résumé—Une analyse théorique de la conduction thermique à travers des couches cylindriques est conduite pour introduire le concept de phénomène de court-circuit de la diffusion. On montre que dans le cas d'une coque à deux couches, des drainages énergétiques sont des caractéristiques de ce phénomène. Un cas limite de solution lorsque la conductivité de la couche la plus faible croît par rapport à ce qui l'entoure, est utilisé pour résoudre le dilemne de la discontinuité des températures de coin dans le cas d'une bande semi-infinie où d'une plaque finie. Une méthode pour adoucir les discontinuités de température est développée sur la base de considérations physiques.

KURZSCHLUSSERSCHEINUNGEN: WÄRMESTROMPFADE IN GESCHICHTETEN ZYLINDRISCHEN WÄNDEN

Zusammenfassung—Es wird eine grundlegende theoretische Untersuchung der stationären Wärmeströmung durch geschichtete zylindrische Wände durchgeführt und dabei das Konzept von Kurzschlußerscheinungen der Wärmeleitung eingeführt. Im Falle einer zweischichtigen Wand zeigt es sich, daß Strompfade mit erhöhter Energiedichte eine hervorspringende Eigenschaft solcher Erscheinungen sind. Eine Grenzbetrachtung der hierbei entwickelten Lösung, insbesondere für den Fall, daß die Leitfähigkeit der unteren Schicht über alle Grenzen steigt, dient dazu, das alte Problem unstetiger Ecktemperaturen im Falle des halbunendlichen beheizten Streifens oder der endlichen Wand zu lösen. Eine Methode zur Glättung der Unstetigkeiten der Temperatur auf einer sinnvollen physikalischen Grundlage wird in dieser Arbeit entwickelt.

ЯВЛЕНИЯ КОРОТКОГО ЗАМЫКАНИЯ. СТОКИ ТЕПЛОВОЙ ЭНЕРГИИ В КОМПОЗИТНЫХ МАТЕРИАЛАХ ЦИЛИНДРИЧЕСКОЙ ФОРМЫ

Аннотация—Выполнен теоретический анализ стационарного теплового потока через композитные материалы цилиндрической формы для исследования явлений переноса при коротком замыкании. В случае двухслойного материала показано, что для таких явлений характерен более интенсивный сток энергии. Предельный случай полученного в работе решения, в особенности при увеличении теплопроводности нижнего слоя до бесконечности, используется для разрешения давно существующей проблемы о наличии разрывов температур в угловых областях нагреваемых полубесконечной полоски или пластины конечных размеров. Разработан физически обоснованный метод сглаживания разрывов непрерывности температур.